4

Hyperbolic mirror

In this chapter, some important properties of the hyperbolic mirror will be mathematically described. This offers results that can be used for further applications like hyperbolic mirror computation and simulation and bird's eye view.

4.1 Hyperbolic Formula

The general hyperbola can be expressed by the formula

$$\frac{x_p^2}{a^2} - \frac{y_p^2}{b^2} = 1, \tag{4.1}$$

where a is the distance from the coordinate system's origin to the intersection points of the hyperbola with the x-axis. The terms $\frac{b}{a}$, resp. $-\frac{b}{a}$ describe the gradients of the two asymptotes, to which the hyperbola converges for $x \to \infty$ and $x \to -\infty$. Additionally, $\varepsilon = \sqrt{a^2 + b^2}$ is the distance f (focal distance) from the origin to the hyperbola's two focal points F and F' (see Fig. 4.1).
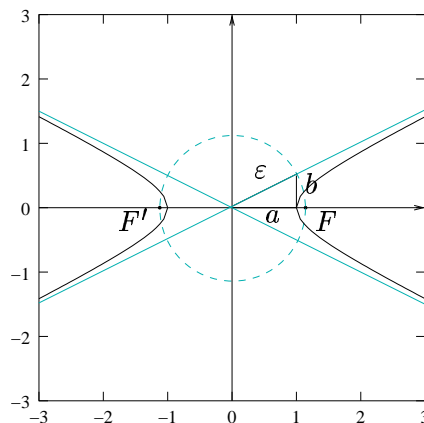

Figure 4.1: General hyperbola with $a = 1$, $b = 0.5$

4.2 Optical Properties of the Hyperbolic Mirror

The properties of a hyperbolic mirror can be derived from the rule $\alpha_{pt} = \alpha_{tr}$, that the angle of incidence is equal to the angle of reflection (Fig. 4.2).
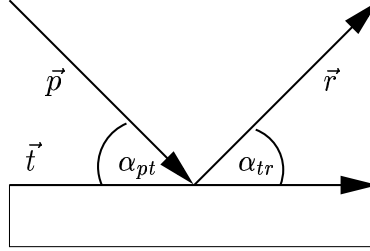

Figure 4.2: Reflection of a light ray on a planar mirror surface

4.2.1 Analysation of the Hyperbolic Formula

For this reason, on the one hand the regular formula of the hyperbola from equation 4.1 has to be analysed

$$\frac{x_p^2}{a^2} - \frac{y_p^2}{b^2} \quad = \quad 1 \quad \Leftrightarrow \quad y_p^2 \quad = \quad \frac{b^2 \cdot (x_p^2 - a^2)}{a^2}$$

$$\Leftrightarrow \quad y_p \quad = \quad \pm \frac{b}{a} \cdot \sqrt{x_p^2 - a^2} \qquad (4.2)$$

The hyperbola's clipping of the first quadrant ($x_p \geq 0 \wedge y_p \geq 0$) already describes the mirror's shape as a rotational entity, therefore equation 4.2 can be simplified

$$y_p \quad = \quad \frac{b}{a} \cdot \sqrt{x_p^2 - a^2} \qquad ; \text{with} \quad x_p \geq 0 \qquad (4.3)$$

Accordingly, the incoming ray's gradient s_p can be determined by

$$s_p \quad = \quad \frac{y_p}{x_p} \quad = \quad \frac{b}{a \cdot x_p} \cdot \sqrt{x_p^2 - a^2} \qquad (4.4)$$

On the other hand, using this restriction yields the hyperbola's tangent

$$\frac{x_t \cdot x_p}{a^2} - \frac{y_t \cdot y_p}{b^2} \quad = \quad 1 \quad \Leftrightarrow \quad y_t \quad = \quad \frac{b^2 \cdot x_t \cdot x_p}{a^2 \cdot y_p} - \frac{b^2}{y_p}$$

$$\Leftrightarrow \quad y_t \quad \overset{(4.3)}{=} \quad \frac{b \cdot (x_t \cdot x_p - a^2)}{a \cdot \sqrt{x_p^2 - a^2}}$$

$$\Leftrightarrow \quad y_t \quad = \quad \frac{b \cdot x_p}{a \cdot \sqrt{x_p^2 - a^2}} \cdot x_t - \frac{a}{\sqrt{x_p^2 - a^2}} \qquad (4.5)$$
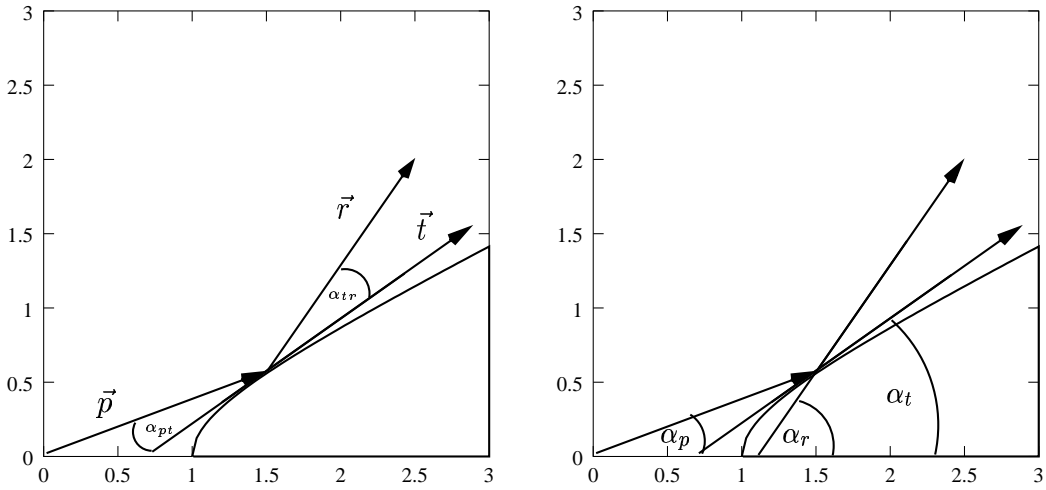
Figure 4.3: Reflection of a light ray on a hyperbolic surface (first quadrant)

Thus, the tangent's gradient s_t can be expressed subject to x_p as

$$s_t \quad = \quad \frac{b \cdot x_p}{a \cdot \sqrt{x_p^2 - a^2}} \tag{4.6}$$

4.2.2 Computation of the Reflection's Angle by using the *atan2*

One way to calculate the angle of reflection α_r is to determine the solution by using the angle of incidence α_p, the tangent's angle α_t and the *arcus-tangens-2* function

$$\begin{aligned} \alpha_r \quad &= \quad 2 \cdot \alpha_t - \alpha_e \\ &= \quad 2 \cdot atan2\left(s_t\right) - atan2\left(s_p\right) \end{aligned} \tag{4.7}$$

Because of the use of the *arcus-tangens* function this solution has multiple disadvantages:

- The gradient s_r of the reflected ray

$$s_r \quad = \quad tan\left(\alpha_r\right) \quad = \quad tan\left(2 \cdot atan2\left(s_t\right) - atan2\left(s_p\right)\right)$$

  which is necessary for further analysis, can hardly be simplified and solved in order to get x_p.

- In the case that the reflection angle is exactly upright, which is equal to $\alpha_r = \frac{\pi}{2}$, the tangens and the gradient to calculate (s_r) are not defined:

$$\lim_{\alpha_r \to \frac{\pi}{2} - 0} \left[tan\left(\alpha_r\right)\right] = \infty \qquad \lim_{\alpha_r \to \frac{\pi}{2} + 0} \left[tan\left(\alpha_r\right)\right] = -\infty$$

- The multiple application of trigonometric functions is more expensive in calculation time than the following approach.

4.2.3 Computation of the Reflection's Angle by using Vector Calculation

The second approach to determine the ray of reflection directly follows from the simple reflection rule that the angle of incidence is equal to the angle of reflection:

$$\alpha_{pt} = \alpha_{tr} \tag{4.8}$$

$$\Leftrightarrow \quad cos(\alpha_{pt}) = cos(\alpha_{tr})$$

$$\Leftrightarrow \quad \frac{\vec{p} \cdot \vec{t}}{\|\vec{p}\| \cdot \|\vec{t}\|} = \frac{\vec{r} \cdot \vec{t}}{\|\vec{r}\| \cdot \|\vec{t}\|}$$

$$\Leftrightarrow \quad \left| \begin{array}{rcl} x_p \cdot x_t + y_p \cdot y_t & = & x_r \cdot x_t + y_r \cdot y_t \\ x_p^2 + y_p^2 & = & x_r^2 + y_r^2 \end{array} \right|$$

$$\Leftrightarrow \quad x_r = \left\{ x_p, \quad \frac{2 \cdot x_t \cdot y_t \cdot y_p + (x_t^2 - y_t^2) \cdot x_p}{x_t^2 + y_t^2} \right\}, \tag{4.9}$$

$$y_r = \left\{ y_p, \quad \frac{2 \cdot x_t \cdot y_t \cdot x_p + (y_t^2 - x_t^2) \cdot y_p}{x_t^2 + y_t^2} \right\} \tag{4.10}$$

The first solution corresponds to the incoming ray $\vec{p}$. Thus, there exists a unique solution for the reflected vector,

$$x_r = \frac{2 \cdot x_t \cdot y_t \cdot y_p + (x_t^2 - y_t^2) \cdot x_p}{x_t^2 + y_t^2}, \tag{4.11}$$

$$y_r = \frac{2 \cdot x_t \cdot y_t \cdot x_p + (y_t^2 - x_t^2) \cdot y_p}{x_t^2 + y_t^2} \tag{4.12}$$

having none of the disadvantages of the first approach.

4.2.4 Calculation Results

By using the formulas from sections 4.2.1 and 4.2.3 the following inputs allow to determine the following output values (see Fig. 4.3):

Input	Output
• Hyperbola parameters a, b • The point of reflection x_p	• Vector of incoming ray $\vec{p} = (x_p \vert y_p)$ • Vector of tangent ray $\vec{t} = (x_t \vert y_t)$ • Vector of reflection ray $\vec{r} = (x_r \vert y_r)$ • Angle of each ray $\alpha = atan2(y, x)$

4.3 Further Applications

4.3.1 Hyperbolic Focal Point

This offers the possibility to get the virtual focal point of the hyperbolic mirror, where all reflections of incoming rays intersect. Thus, if all reflections intersect in only one point, the picture taken by the camera will be sharp in all regions.
By defining a shift Δx_p along the x-axis

$$x_g(x_p) = x_p + \Delta x_p \tag{4.13}$$

and determining the x-intersection x_{r0} of a reflected ray x_r (see Fig. 4.4)

$$x_{r0}(x_p) = x_g(x_p) - \frac{x_r(x_p)}{y_r(x_p)} \cdot y_p \tag{4.14}$$

this problem can be solved by variation calculation, which means to find a solution for

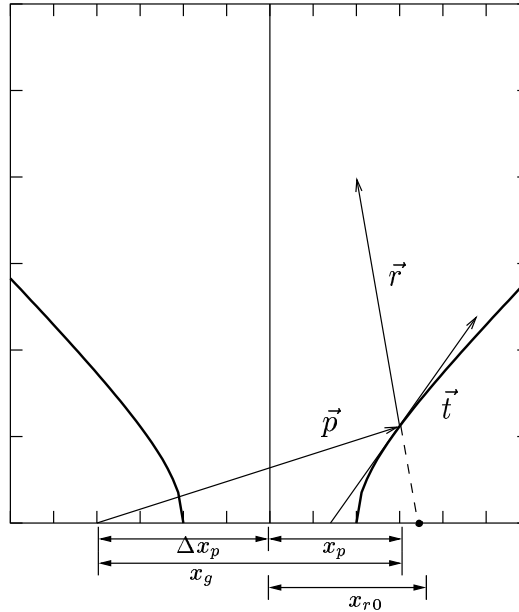$$x_{r0}(x_p) = x_{r0}(x_g(x_p)) \qquad \forall x_p \tag{4.15}$$

Figure 4.4: Extensions for determining unique focal point

There can be found two solution for this equation subject to Δx_p:

$$\Delta x_p = \left\{ \sqrt{a^2 + b^2}, \quad -\sqrt{a^2 + b^2} \right\} \tag{4.16}$$

For the cause that the first one is equal to the hyperbola's focal point (see section 4.1), the second one is the unique solution.

Thus, A sharp picture can only be taken, if the focal point of the camera exactly lies in F' (see section 4.1), opposed to the hyperbola's focal point.

This is the main advantage of hyperbolic mirror shape, which is not supported by spherical mirrors, for example.

4.3.2 Hyperbolic Mirror Simulator

Using the formulas derived in the last sections, a program was developed to simulate reflections on several hyperbolic mirror shapes.
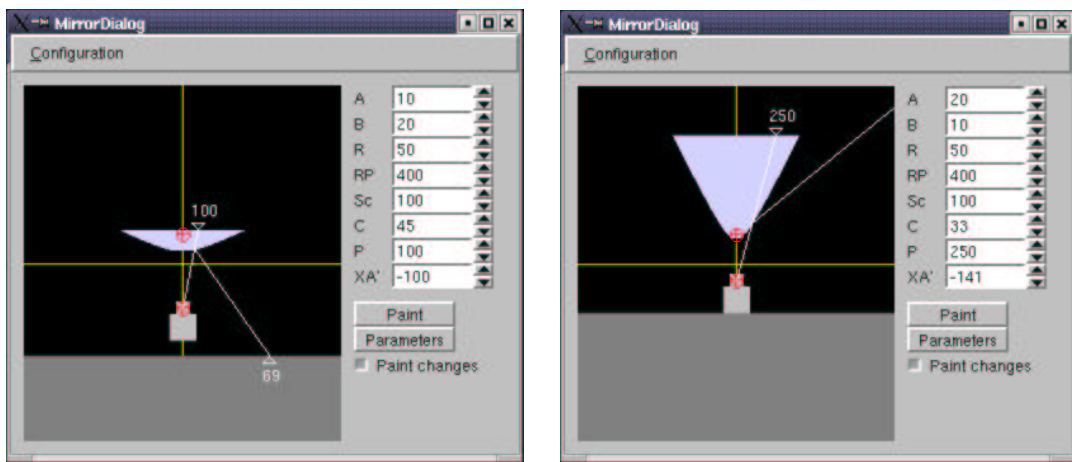

Figure 4.5: Two screenshots of the MirrorDialog to simulate a hyperbolic mirror.

Like it can be seen in Fig. 4.5, there is a graphical visualisation of the system, which is changing dynamically when changing any of the parameters besides. These are:

a, b	hyperbolic parameters
r	real upper radius of the hyperbolic mirror
rp	upper radius of the hyperbolic mirror in the round image
sc	scaling factor to zoom in / out
c	distance from camera focal point to the bottom of the mirror
p	regarded point in round image (distance to round image center)
xa	distance from floor to top of the mirror

Table 4.1: Parameters for MirrorDialog.

Additionally, a virtual light ray can be send out from the camera focal point towards the mirror by setting p. An output of several informations can be reached by clicking the button *„Parameters"*. These are the internal parameters, further computed

distances and determined values for the incoming, the tangent and the reflected ray
(depending on p). The following is an example printout for the left mirror from Fig. 4.5.

```
********** Internal Parameters ***************
A  : 10.00000 (first parameter of hyperbolic)
B  : 20.00000 (second parameter of hyperbolic)
R  : 50.00000 (upper radius of hyperbolic)
RP : 400.00000 (upper radius of hyperbolic [Pixel])
Sc :  1.00000 (scaling factor)
C  : 45.00000 (distance from camera F to bottom of hyperbolic)
C1 :  -35.00000 (distance from camera F to hyperbolic 0|0)
F  : 22.36068 (distance from hyperbolic F to hyperbolic 0|0)
H  : 26.92582 (height of mirror (from hyperbolic 0|0))
H2 :  16.92582 (height of mirror)
XAV: -100.00000 (floor distance to upper hyperbolic)
-----------------------------------------------
Px :  12.50000 (x-coordinate of point P (image))
Py :  61.92582 (y-coordinate of point P (image))
Px2:  9.29061 (x-coordinate of point P (mirror))
Py2:  11.02628 (y-coordinate of point P (mirror))
Pa :  78.58801 (angle of vector of point P (mirror))
-----------------------------------------------
Tx :   4.74728 (x-coordinate of tangent (mirror))
Ty :   1.00000 (y-coordinate of tangent (mirror))
Ta :   11.89528 (angle of vector of tangent (mirror))
-----------------------------------------------
Rx :   36.41836 (x-coordinate of reflection (mirror))
Ry :  -51.62132 (y-coordinate of reflection (mirror))
Ra :  -54.79744 (angle of vector of reflection (mirror))
YA : 24.19528 (y-axis intercept of reflection)
XA : 68.62268 (floor axis intersect of reflection)
***********************************************
```

Each configuration can be saved and reloaded in the *„Configuration"* menu.

4.3.3 Bird's Eye View

Having correctly put in the system parameters for a real system, a bird's eye view
can be calculated from the round image. Like the panoramic image is the projection
of the round image onto a cylindrical shape, the bird's eye picture is the projection
onto a plane, usually the ground plane.
By using configuration information from the MirrorDialog (see section 4.3.2), it is
possible to assign each image pixel to a point on the floor.

In the test environment (see Fig. 3.4), it is possible to extract the floor-wall edges by
edge detection. By projecting each edge point to the floor by bird's eye view, the real
distances to the walls can be estimated.

The following experiment was made with the mirror shape described in section 3.2 and a self-made calibration board, results can be seen in Fig. 4.6.
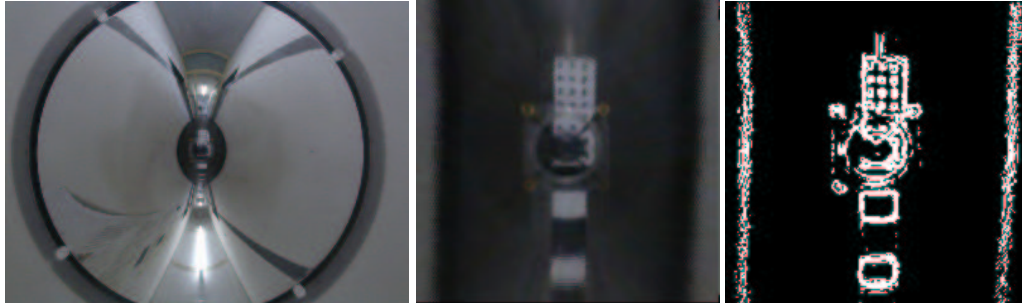

Figure 4.6: Left: Original round image (1600x1200). Center: Bird's eye view (201x201). Right: Bird's eye view of thresholded edge image (201x201).

A Line-Hough-Transformation method adapted on the bird's eye image has been implemented to get information about lines in the image. Each line can be described by only two parameters l and a, equal to length and angle of the line's normal from the respective line to image center. These information are accumulated in a partitioned 2-dimensional space. Accumulator cells that have collected most points after having processed the image are supposed to be the image's most important lines.

Using the binary bird's eye view image, the transformation offers the two main lines shown in Tab. 4.2, where c is the pixel count of the accumulator cell.

	c	a	l
Line 1	133	357	88
Line 2	134	180	88

Table 4.2: Hough-Transformation results for the binary bird's eye view from Fig. 4.6.

Note that the pixel distances l are equal to real distance estimates (here in cm) that can be determined by this method. Regarding that the image has been taken at the middle of the floor, which corresponds to a distance of 82 cm to each wall, the resulting values of 88 cm are very good.